

# The Emergence of Probabilities in Anhomomorphic Logic<sup>1</sup>

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## Abstract.

Anhomomorphic logic is a new interpretation of Quantum Theory (due to R. Sorkin). It is a *histories* formulation (c.f. consistent histories, quantum measure theory). In this approach, reality is a co-event, which is essentially an assignment of a truth value {True, False} to each question. The way this assignment is done mimics classical physics in as much as possible, allowing however for sufficient flexibility to accommodate quantum ‘paradoxes’, as is shown by the analysis of Kochen-Specker theorem. In this contribution, after briefly reviewing the approach, we will examine how probabilistic predictions can arise. The Cournot principle and the use of approximate preclusions will play a crucial role. Facing similar problems in interpreting probability as in classical probability theory, we will resort to the *weak* form of Cournot principle, where possible realities will be preclusive co-events and the quantum measure is used to obtain predictions. Examples considered, includes the fair coin and the double slit pattern arguably one of the most important paradigms for quantum theory.

## 1. Introduction

Quantum theory challenges the picture we had in classical physics about what reality is. In order to retain the classical picture alternative formulations, such as hidden variables, were proposed, facing however serious problems due to the severe restrictions from observations that result from Bell’s inequalities [1] and the Kochen-Specker theorem [2]. Quantum theory (as it stands now) requires some external (to the system) observer, in order to “make sense”. However, for the needs of quantum cosmology for example, where we have a truly closed system,

<sup>1</sup> Based on talk given by P. Wallden at the DICE 2008 conference

we do lack an interpretation. Consistent histories [3, 4, 5], can be seen as an attempt to deal with this, that also removed in some way the very special role that time has in standard quantum theory. The latter is another desired feature, if one is interested in building a quantum theory of gravity where time is in same footing with space. Consistent histories however, failed to provide a fully satisfactory interpretation of the quantum theory of closed systems due to the context dependence ( or else the dependence of predictions on the consistent set realized, which arises due to the fact that there exist many incompatible consistent sets). As a development came a novel interpretation which retains essentially the same mathematical structure (history space and decoherence functional) the so-called, *anhomomorphic logic* <sup>2</sup> [6, 7]. In the next section we introduce this approach briefly, before we come back to the core of this contribution, which is how probabilistic predictions are dealt within this approach<sup>3</sup>.

## 2. Introducing Anhomomorphic Logic

### 2.1. Classical Physics, Histories and Logic

Let us first consider classical physics, in which we use Histories and (classical) Logic in order to ask questions about nature.

We have a set  $\Omega$  (call it *History Space*) of all possible histories  $h_i$ . We can think of each  $h_i$  as a trajectory (or more generally a full specification of the particle in every moment of time). In classical physics one and only one of these histories is actually realized. However, in stochastic physics (non-deterministic) we do not know which one is realized, but have a probability measure on  $\Omega$ <sup>4</sup>. All possible questions about a system can now be rephrased in terms of asking whether the real history belongs to some subset of  $\Omega$ . So for example if we want to ask whether the particle is at the interval  $\Delta$  at time  $t$ , we ask if the realized history  $h$  belongs to the subset  $\Delta_t := \{h_i | h_i(t) \in \Delta\} \subset \Omega$ .

Associated with  $\Omega$  is its power set  $\mathcal{U}$  (set of subsets of  $\Omega$ ), that has a Boolean algebra structure with intersection as multiplication and symmetric difference ( $A \triangle B := (A \cup B) \setminus (A \cap B)$ ) as addition.

We also have the set of truth values  $\mathcal{T}$  ( e.g.  $\{\text{True}, \text{False}\}$ ) which also has a Boolean algebra structure (that of  $\mathbb{Z}_2$ , identifying ‘True’ with 1 and ‘False’ with 0). Finally we have the possible valuations  $\phi_i$ . A valuation,  $\phi_i$  is an assignment of a truth value to each question, i.e. in other words a map from  $\mathcal{U}$  to  $\mathcal{T}$ . We moreover require that this valuation respects the Boolean structures of  $\mathcal{U}$  and  $\mathcal{T}$  by being a homomorphism:

$$\begin{aligned}\phi(A \triangle B) &= \phi(A) + \phi(B) \\ \phi(A \cap B) &= \phi(A)\phi(B)\end{aligned}\tag{1}$$

It can be shown that if one requires the maps  $\phi$  to be homomorphic there is a one to one correspondence between these homomorphic maps and single histories, i.e. each homomorphism corresponds to a characteristic map of a history:

$$\phi_i | \phi_i(A) = 1 \text{ iff } h_i \in A\tag{2}$$

<sup>2</sup> Also known with different names as for example “co-event interpretation”.

<sup>3</sup> More details can be found in Ref. [8].

<sup>4</sup> Histories that have measure zero, are never realized, and thus are not even possible realities. This will be of use later.

## 2.2. Consistent Histories and Quantum Measure

In quantum theory the above picture does not hold. The only fully developed histories formulation (at present) is the consistent histories approach (also known as decoherent histories) Ref. [3, 4, 5]<sup>5</sup>. The basic elements are histories of the system and to each pair of histories a complex number is assigned by the means of the *decoherence functional* and it corresponds to the interference between these histories. It is defined as:

$$\begin{aligned} D(A, B) &= D^*(B, A) \\ D(A \sqcup B, C) &= D(A, C) + D(B, C) \\ D(A, A) &\geq 0 \\ \sum_{h, h'} D(h, h') &= 1 \end{aligned} \tag{3}$$

The diagonal elements correspond to *candidate* probabilities and under certain circumstances<sup>6</sup> we can interpret this (real) number as the probability of this history (or in general coarse grained history i.e. subset of histories) actually occurring (this probability is conditional on the classical domain, or consistent set, that is actually realized). For more details the reader is referred to the original references [3, 4, 5].

This candidate probability interpretation led R. Sorkin to think of the diagonal elements of a decoherence functional as a *quantum measure*, as they mimic some of the properties of a classical measure, and can be thought of as the first step in a chain of generalizations of classical measure theory (see Refs. [9, 10]). We will use the notation  $\mu(A) := D(A, A)$  for the quantum measure. While it is normalized to unity and is positive-definite, it fails to obey the “additivity of disjoint regions of the sample space”, a necessary requirement to have a probability measure,

$$\mu(A \sqcup B) \neq \mu(A) + \mu(B) \tag{4}$$

This is due to interference (c.f. double slit). Note though that the quantum measure *does* obey the weaker

$$\mu(A \sqcup B \sqcup C) = \mu(A \sqcup B) + \mu(A \sqcup C) + \mu(B \sqcup C) - \mu(A) - \mu(B) - \mu(C) \tag{5}$$

Which means that in quantum theory, there is no fundamentally new three paths interference. In particular, it can be shown that any quantum measure that obeys Eq. (5) can arise from some decoherence functional (as defined above, and thus the equivalence between quantum measures and decoherence functionals.).

Due to the difference of the quantum measure compared to classical measure, the picture of classical histories and logic analyzed in the previous section, cannot be retained. The most striking problem, comes essentially from the Kochen-Specker theorem, and is discussed in Ref. [11]. A simplified version form of this problem comes from the three slit experiment, where we have  $\mu(A \sqcup B) = \mu(B \sqcup C) = 0$  but  $\mu(A \sqcup C) \neq 0$ . The first two imply that all the set  $\{A, B, C\}$  is impossible (since it is covered by measure zero sets), however one subset,  $\{A, C\}$ , *is* possible.

<sup>5</sup> Mathematically is very close to the well known sum-over-histories formalism of Feynman, but the interpretation is quite different

<sup>6</sup> If it belongs to a partition that the elements pairwise have zero interference.

### 2.3. Anhomomorphic Logic

We can see now that in order to be compatible with quantum theory, one needs to alter something in the classical picture. Anhomomorphic logic is the approach where we maintain the same set of possible questions (namely  $\mathcal{U}$ ), same possible truth values ( $\mathcal{T} = \{\text{True}, \text{False}\}$ ) but we change the allowed valuation maps,  $\phi_i$ , by weakening the requirement the map is a homomorphism. These maps (no longer homomorphic) are called *co-events*. This approach was initiated by R. Sorkin in Refs. [6, 7]. As already stated, F. Dowker and Y. Ghazi-Tabatabai in Ref. [11] showed how the suggested approach evades Kochen-Specker theorem<sup>7</sup> and thus is a good candidate for a realistic interpretation of quantum theory.

While we want to weaken the requirement to be homomorphism we need it to maintain sufficient structure to be able to make deductions (deductive logic), retain some sense of reality and in the same time be able to accommodate the paradoxes of quantum theory. There are three conditions we require.

- (i) We retain the preservation of multiplication under  $\phi$  but no longer require the preservation addition in general,

$$\begin{aligned}\phi(A \cap B) &= \phi(A)\phi(B) \\ \phi(A \triangle B) &\neq \phi(A) + \phi(B)\end{aligned}\tag{6}$$

This is called *multiplicative* co-event<sup>8</sup>. It has the following desirable properties.

- (a)

$$\phi(A) = 1 \text{ and } A \subset B \Rightarrow \phi(B) = 1\tag{7}$$

This is the basic inference rule (“modus ponens”), and can be used to make deductive proofs<sup>9</sup>. However we *cannot* use proofs by contradiction (excluded middle), since

$$\phi(A) = 0 \nRightarrow \phi(\Omega \setminus A) = 1\tag{8}$$

This is the case in intuitionistic logic also (see also constructive mathematics for mathematics using only deductive proofs). Note though, that the following *does* hold:

$$\phi(A) = 1 \Rightarrow \phi(\Omega \setminus A) = 0\tag{9}$$

- (b) It can be shown, that each multiplicative co-event<sup>10</sup>  $\phi_A$  corresponds to a subset  $A$  called *dual* of the co-event<sup>11</sup>, such that

$$\phi_A(B) = 1 \text{ if and only if } A \subseteq B\tag{10}$$

This is similar to the case at classical physics, where the duals were single element subsets (corresponding to characteristic maps). For convenience we will occasionally identify the co-events with its dual.

<sup>7</sup> See also Ref. [12] for related work.

<sup>8</sup> Earlier alternative definitions were use as in Ref. [6] but this turned out to be the most satisfactory due to several reasons.

<sup>9</sup> e.g. “I am physicist”-True, along with “Physicists are humans”-True, implies that “I am human”-True.

<sup>10</sup> at least for finite dimensional history space  $\Omega$  but similar considerations can be generalized after taking care.

<sup>11</sup> In literature, the dual is also referred occasionally as the “suport”.

- (ii) We require that every subset of measure zero is always mapped to zero (i.e. it is always false).

$$\mu(A) = 0 \Rightarrow \phi(A) = 0 \quad (11)$$

Co-events obeying equation 11 are called *preclusive*. This requirement is same as in classical physics, where the histories that are (or belong to a set) of measure zero cannot occur, in other words are not possible realities.

- (iii) We require that the co-event (or actually its dual) is *primitive*. This means the following: We (partially) order all possible duals (which are simply subsets of  $\Omega$ ) with respect to set inclusion. Then from all the (multiplicative) preclusive co-events we choose only those that are minimal with respect to this order, i.e. the finest grained (smaller) duals. Remember that in classical physics, these are simply single element subsets ( $h_i$ ), so by requiring primitivity we come as close as possible to the picture we have in classical physics.

We therefore get to the point where we identify these co-events, which are (multiplicative), preclusive and primitive, as possible realities (PPC). The multiplicativity, allows us to view the relevant duals as what actually happens, and thus we have a very similar picture with classical physics, where the difference is that reality is no longer a fine-grained description but rather a coarser-grained one<sup>12</sup>. A very interesting thing to point out here, is that the logic that arises, depends on the dynamics (and initial condition), via the use of the quantum measure (the zero sets) and it is *not* fixed a-priori.

Before moving to the main text of this contribution, we shall stress one more interesting point. Classical physics corresponds to homomorphisms while the “quantum” nature is encoded in the parts of the map that are anhomomorphic. Thus we could say that a classical domain arises if we consider some coarse graining (i.e. some subalgebra of  $\mathcal{U}$ ) such that the induced map on this subalgebra, is a homomorphism for ANY of the allowed (PPC) co-events. What is most interesting, is that with this notion of a classical domain there exists a unique finest grained classical domain that all the others arise as coarse-grainings. This is in striking difference with consistent histories where the main problem was the existence of many incompatible classical domains. The reader is once again referred to Ref. [8] for details.

### 3. Probabilities in Anhomomorphic Logic

#### 3.1. Probabilities in Classical Closed Systems: Cournot Principle

In the analysis we have done so far, we have argued about what a possible reality is (a PPC co-event), but have omitted what is probable. In most everyday life cases, the predictions we get (particularly from quantum theory) are probabilistic. However here we shall recall that in consistent histories we have a single closed system, and in such cases the very concept of probability (in classical physics) is not easily and unambiguously defined. For example a probabilistic statement about the state of the full universe cannot be testable, since either outcome (finding universe possessing the property in question or not) would not falsify our initial assertion.

There was a big philosophical debate by the founders of probability theory on how one is expect to understand a probabilistic statement<sup>13</sup>. The standpoint which we shall adopt (which suits best the case of a single closed system such as the universe) is the use of the *Cournot Principle*, and is closely related with experimentally falsifying a theory :

<sup>12</sup> We can view this, as either finer grained questions are un-physical, or physics at finer grained scales has some (classically) contradictory properties.

<sup>13</sup> See for example Kolmogorov in the “Grundbegriffe”.

**(Strong) Cournot Principle:** In a repeated trial, an event  $A$  of small measure ( $\epsilon$  arbitrarily chosen), does *not* occur.

Note that in a repeated trial, distributions of outcomes that differ drastically from the probability distribution of the single trial, would have very small measure and thus they do not occur. However, this picture is problematic in classical physics. To see this, we can consider the case of tossing a fair coin. The probability measure of having all heads in  $N$  trials, goes as  $1/2^N$ , while the probability of getting heads less than (say) 60% is  $\sum_{k=1}^{0.6N} \binom{N}{k}$  which is much greater than  $1/2^N$ . We could thus claim that an outcome of all heads is impossible, while an outcome with 50% heads is possible. However, we note, that in any actual realization (series of outcomes), we will get a sequence of results (e.g. *htthththt...*) that has probability of occurring  $1/2^N$  exactly the same with the one we get with only heads. This probability was very small ( $< \epsilon$ ) and thus this outcome was prohibited. In this example, we note that *any* possible outcome has measure less than  $\epsilon$  and if we take the (strong) Cournot seriously that would come to a direct contradiction with reality (since something actually happens). Such considerations lead us to a milder version of the Cournot principle:

**Weak Cournot Principle:** In a repeated trial, an event  $A$  *singled out in advance*, of small measure ( $\epsilon$  arbitrarily chosen), will *not* occur.

This means that if we ask in advance: “Is the outcome 50% heads possible” we will get the answer yes. However, if we ask in advance “will the sequence *htthththt* occur” we will get the answer no, since the measure for this outcome is small. We should stress here that there is a split between the ontology and predictability of the theory. According to weak Cournot, any outcome that doesn’t have identically zero measure, is possible. However if we have a theory and we ask a question that has small ( $< \epsilon$ ) non-zero measure, if this outcome occurs, we falsify our theory, even if our theory did not exclude this event from occurring. For example, if  $N$  coin tosses give  $N$  heads, we falsify our assertion that the coin was fair, even if it is conceivable that it was a fair coin. To sum up, in strong Cournot, no approximately zero measure set is a possible reality, while in weak Cournot, approximately zero measure sets are possible realities. In order to make scientific predictions, we need to preselect questions and we may need to falsify our theory even if the experimental outcome is (ontologically speaking) a possible reality.

Coming back to the case of a closed *single* system, note that while we cannot make a probabilistic statement for the universe itself, we can instead make probabilistic predictions for subsystems that ‘look identical’. In the above example we made statements about the distribution of outcomes of tosses of coins (subsystems). These statements, if we view them as statements about coins, correspond to non-trivial probabilities. However, they correspond to ‘almost trivial’ (i.e. almost zero or almost one) probabilities for the full system of  $N$  coins. The reader is referred to Ref. [8] for further details.

### 3.2. Quantum Theory

One of the things that stopped us from taking the strong Cournot seriously, was the fact that we had an example that all  $\Omega$  was covered by histories (possible outcomes,  $h_i$ ’s<sup>14</sup>) each of which was precluded because it had measure  $\mu(h_i) < \epsilon$ . This resembles the Kochen Specker theorem. Anhomomorphic logic, by considering as possible realities subsets of  $\Omega$  other than singletons (fine-grained histories  $h_i$ ), had evaded this contradiction. One would hope that it may be possible to take strong Cournot seriously in the quantum case, by defining Approximate

<sup>14</sup> Note that by  $h_i$  here we mean a sequence of single outcomes, as for example *htthththt* in the coin case.

PPC (APPC) co-events by replacing the condition ‘ $\mu(A) = 0$ ’ with the condition ‘ $\mu(A) < \epsilon$ ’ in the definition of preclusion. (c.f. with Eq.11). We hope that by making this change we can encode all the ‘useful’ information in the quantum measure in our set of APPC co-events; then the quantum measure itself would no longer be needed for predictions. However this leads to contradictions with observation. For if the dual of a co-event contains two histories giving different answers to a particular question, then the co-event itself will answer our question in a manner inconsistent with Boolean logic; even when this question relates to an experimentally observable outcome (see Ref. [8] for further details). For example, returning to our repeated trial of a coin, assume we have a APPC co-event that has in its dual one history that has a heads outcome in the first coin toss, and a second history that has a tails outcome. Then by the definition of anhomomorphic logic, we would get NO to the question “did the first coin toss result in heads?” and also NO to the question “did the first coin toss result in tails”. But we can experimentally verify that exactly one of the two outcomes ‘heads’, ‘tails’ will actually occur if we are to throw our coin. Such co-events (that are possible for APPC co-events), would not be possible had we stick with *exactly* preclusive co-events (see Ref. [8] for further details). We are then pushed back to the use of the weak Cournot.

In quantum theory when we use weak Cournot, the possible realities are still the PPC co-events (c.f. in classical physics, possible realities were fine grained histories, that had non-zero measure, even if it was arbitrarily small). To deduce probabilistic predictions we need to use once again the quantum measure. We select a question corresponding to a subset  $A$  of the history space  $\Omega$ . If  $\mu(A) < \epsilon$ , we say that this result is not possible, and this is the prediction we make. If this outcome actually arises, we then say that our initial assumption about what was the quantum measure (which includes initial conditions and dynamics) has been falsified.

Note, the following difference of the quantum case compared to the classical one. In classical physics the measure that is used to derive predictions, is on the space of possible realities, which is the history space. In quantum theory, the quantum measure is again on the history space, however this is no longer the space of possible realities, since the possible realities are the PPC co-events. The reader is referred to Ref. [8] for further details.

### 3.3. The Double Slit Example

Finally, let us show how the quantum measure can be used along with weak Cournot to get predictions that reproduce the double slit pattern. For simplicity we will consider a discrete screen, with 5 slots ( $i \in \{0, \pm 1, \pm 2\}$ )<sup>15</sup>. We also have two slits ( $s_1, s_2$ ). Each fine grained history (for a single repetition), consists from the particle crossing one slit and hitting the screen at on slot. The measure is given:

$$\begin{aligned} |(s_j, i)| &= 0.1 \\ |(s_1, i)| \sqcup |(s_2, i)| &= \begin{cases} 0.3 & \text{if } i = \pm 2, 0 \\ 0.05 & i = \pm 1 \end{cases} \\ |(s_k, i) \sqcup (s_l, j)| &= |(s_k, i)| + |(s_l, j)| \\ &= 0.2 \text{ if } i \neq j \end{aligned}$$

If we have many independent repetitions we can simply use the product measure. Now assume that we have a closed system with 10 particles that are to cross this double slit, and we assume that anything that has measure less than  $\epsilon = 10^{-3}$  is precluded, meaning that if such

<sup>15</sup> Slots 0 and  $\pm 2$  are the bright fringes and  $\pm 1$  the dark fringes of the pattern.

outcome comes we falsify our initial theory (according to the weak Cournot). We can easily see that the question “Is the distribution uniform” consists of all the permutations where we have 2 particles at each slot, and this number is  $\binom{10}{2}\binom{8}{2}\binom{6}{2}\binom{4}{2} = 113400$  which is much larger than the number of permutations for getting the double slit pattern (i.e. 3 particles at  $\pm 2, 0$  slots and 1 at either 1 or -1). The latter is 4800. However the measure for each fine grained history of 10 particles that are uniformly distributed is only  $\approx 5 \times 10^{-9}$ . The total measure of all histories that are of uniform distribution is then  $113400 \times 5 \times 10^{-9} \approx 5 \times 10^{-4}$ . This is less than  $\epsilon$ , so the uniform distribution is precluded. In contrast while there are much fewer combinations that give the double slit distribution, the measure for each of them is  $\approx 10^{-6}$  and thus the total measure for getting a distribution ‘like’ the double slit pattern is  $4800 \times 10^{-6} \approx 5 \times 10^{-3}$ , which is greater than  $\epsilon$  and thus the double slit distribution is not ruled out by our theory<sup>16</sup>. Here, in this simplified model, we see how predictions of the type of double slit pattern (i.e. ruling out distributions such as the uniform one) can arise in anhomomorphic logic, when we use the quantum measure of many repetitions of the system along with the weak Cournot principle.

#### 4. Summary and Conclusions

In this contribution, we reviewed the anhomomorphic logic approach to quantum theory, which is a development of the consistent histories approach. Reality is no longer a single fine grained history, but a primitive preclusive multiplicative co-event, which can also be viewed as a coarse grained history. The Kochen Specker theorem is evaded, as is the problem of many incompatible classical domains faced by consistent histories. The core of this contribution was how to deal with probabilistic predictions in this formalism. We resorted to the Cournot principle to give meaning to probabilistic statements and essentially in a frequentist’s view on probability (rather than propensity). The use of strong Cournot principle was ruled out (alas for different reasons than in classical physics). The weak Cournot, introduces a split of ontology and predictions. In classical physics this leads to:

- (a) Ontology: Fine grained histories are the possible realities. In other words one and only one history is actually realized.
- (b) Predictions: The (classical) measure on  $\Omega$  is used, in order to make predictions, that are of the type “if  $A$  is realized and  $\mu_c(A) \leq \epsilon$  then our initial assumption is rejected”.

In quantum theory the picture is similar:

- (a) Ontology: Possible realities are the multiplicative, primitive and (exactly) preclusive co-events. If we view the dual picture (the duals), we can say that what is realized is a coarse grained set of histories or else a non-trivial subset of  $\Omega$ .
- (b) Predictions: The *quantum* measure on  $\Omega$  is used, in order to make predictions, that are of the type “if  $A$  is realized and  $\mu(A) \leq \epsilon$  then our initial assumption is rejected”. Note however, the fact that the quantum measure is on the history space, which in the quantum case, is no longer the space of possible realities.

<sup>16</sup> Note, however, that if we asked more detailed question, such as “the first particle will hit slot 0, the second slot 2, etc” even if the distribution was correct the measure would be very small ( $\approx 10^{-6}$  here) and thus ruled out by our predictions. The “paradox” here, as in classical physics, would be that the eventual outcome of the experiment, would be one of those ruled out histories, and the contradiction is evaded by the crucial concept of *pre-selected* questions, in the definition of weak Cournot.



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